Transformations of surfaces with DP Graph and *DERIVE*

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Translations

Let $g(x,y,z) = f(x-a, y-b, z-c)$

$g(x.y,z) = 0$ is the equation of the surface you receive if you translate the surface, *f*, having equation $f(x,y,z) = 0$ along the vector (a, b, c)

Proof: $(x,y,z) \in f \Leftrightarrow f(x,y,z) = 0 \Leftrightarrow f((x+a)-a,(y+b)-b,(z+c)-c) = 0 \Leftrightarrow$

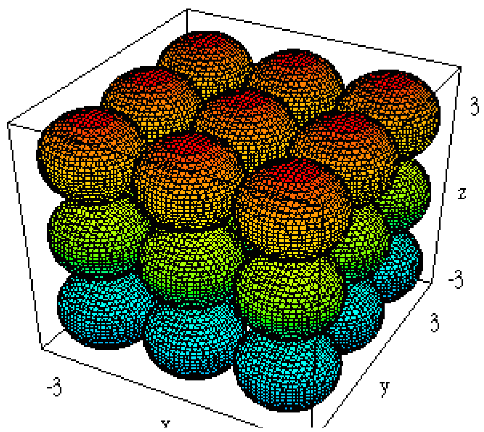
$g(x+a, y+b, z+c) = 0 \Leftrightarrow (x+a, y+b, z+c) \in g$

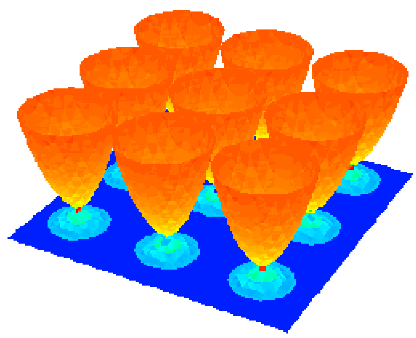
Example: Plot the surfaces you receive if you translate the surface $x^2 + y^2 = \ln(z+3.2)$ along the vectors $(\pm 1, 0, 0)$.

graph3d(((x+2)^2+y^2=ln(z+3.2)^2,(x-2)^2+y^2=ln(z+3.2)^2))

Exercise:

If you create translations of the unit sphere and the surface $x^2 + y^2 = \ln(z+3.2)$ you may obtain the below figures.





Hint: The above 27 spheres can be regarded as one single surface that is the union of 27 spheres.

The equation is

$$\prod_{c=-1}^{1} \prod_{b=-1}^{1} \prod_{a=-1}^{1} \left((x - 2a)^2 + (y - 2b)^2 + (z - 2c)^2 - 1\right) = 0$$

You can use *DERIVE* to generate the above equation.

Choose Multiplication Operator: Asterisk in *DERIVE*. Then simplify the above expression. Copy the result into the Author line and use Ctrl+C to copy the resulting equation to DP Graph.

A general result on transformations of surfaces in implicit form

Definition: A transformation *T* on $\mathbb{R}^3$ is a map $T:\mathbb{R}^3 \to \mathbb{R}^3$.

Let *T* be a transformation on $\mathbb{R}^3$ such that T^{-1} exists. T^{-1} is a transformation such that $T(T^{-1}(x,y,z)) = T^{-1}(T(x,y,z)) = (x,y,z)$ for all $(x,y,z) \in \mathbb{R}^3$.

Theorem: Let $f(x,y,z)=0$ be the equation of a surface, *f*, and let the surface *g* have the equation $g(x,y,z) = f(T^{-1}(x,y,z)) = 0$.

g is the surface you receive if you apply *T* on all the points of the surface *f*.

Proof: $T(x,y,z) \in g \Leftrightarrow g\left(T(x,y,z)\right)=0 \Leftrightarrow f\left(T^{-1}(T(x,y,z))\right)=0 \Leftrightarrow f(x,y,z)=0 \Leftrightarrow (x,y,z)\in f$.

Example: Let *T* be a translation along the vector (a, b, c) then $T(x,y,z)=(x+a, y+b, z+c)$ and

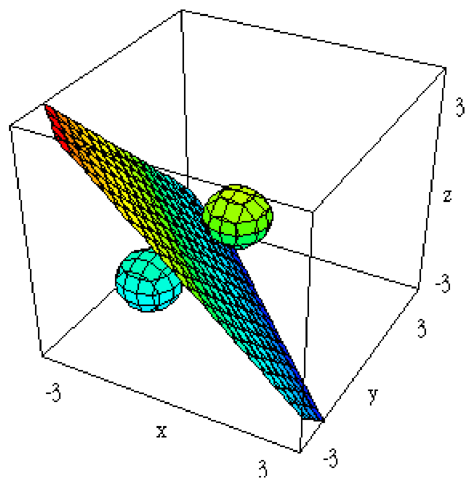
$T^{-1}(x,y,z)=(x-a, y-b, z-c)$. So if you apply the theorem on a translation you receive what we derived on page 1.

Reflections in a plane

We will see how to find the coordinates of the point you receive if you reflect the point (x_0, y_0, z_0) in a given plane, $ax+by+cz+d=0$.

(a,b,c) is a normal vector of the plane. Hence $[x_0+at, y_0+bt, z_0+ct]$ is the normal through (x_0, y_0, z_0) to the plane. Compute the *t*-value, $t=t_0$, of the intersection point between the normal and the plane. Now $(x_0+a2t_0, y_0+b2t_0, z_0+c2t_0)$ are the coordinates of the reflected point. We strongly advice you to use *DERIVE* for these calculations. A reflection is a transformation that is equal to its own inverse. Therefore if you replace *x*, *y* and *z* in $f(x, y, z) = 0$ with the expressions for the coordinates of the reflected point, you will receive the equation of the reflected surface.

In the file refl.dpg we have implemented the above ideas. We have reflected the sphere $x^2+y^2+z^2=0.5$ in the plane $ax+by+cz=d$. The values of *a*, *b*, *c* and d can be varied if you use the scrollbar.



If you like matrices you could write the reflection in matrix form.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow A \begin{bmatrix} x \\ y \\ z \end{bmatrix} + B \,,$$

where $A = \dfrac{1}{a^2+b^2+c^2} \begin{bmatrix} -a^2+b^2+c^2 & -2ab & -2ac \\ -2ab & a^2-b^2+c^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{bmatrix}$ and

$$B = \frac{-2d}{a^2+b^2+c^2} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

If we call the transformation given by our reflection T we have $T^2 = id_{\mathbb{R}^3}$ which means that $T^2(x) = T(Tx)) = x$ for all $x \in \mathbb{R}^3$. You could verify this by computing

$A\left(A \begin{bmatrix} x \\ y \\ z \end{bmatrix} + B \right) + B$. The result of this computation is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

It is a true pleasure to do the above matrix computation with *DERIVE*!

Rotations

(x, y) is a point in $\mathbb{R}^2$. (x', y') is the point you receive if you rotate (x, y) the angle θ around origin. This means that you receive the following equality for complex numbers.

$x' + iy' = (x+iy)e^{i\theta}$ which is equivalent to

$(x', y') = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$ or in matrix form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Now it is straightforward to write down the following formulas for rotation of a point $(x, y, z) \in \mathbb{R}^3$ around the x-axes, y-axes and z-axes respectively

$$R_x(\theta) : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$R_y(\theta) : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$R_z(\theta) : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

We identify the corresponding matrices with $R_x(\theta)$, $R_y(\theta)$ and $R_z(\theta)$

Example: $R_z(\pi)(1,0,0) = (-1,0,0)$ and $R_y\left(\frac{\pi}{2}\right)(0,0,1) = (-1,0,0)$

Example: Derive the equation $g(x,y,z)=0$ of the surface you obtain if you rotate the surface with equation $f(x,y,z)=0$ the angle θ around the *x*-axes.

The map $R_x(-\theta)$ is the inverse map to $R_x(\theta)$. Hence $g(x,y,z) = f\left(R_x(-\theta)\right)(x,y,z)$

The equation of the surface is

$f(x,\ y\cos(-\theta)\ -\ z\sin(-\theta),\ z\cos(-\theta)\ +\ y\sin(-\theta))$=0

Example: Plot the surface you obtain if you rotate $z = x^2 + y^2, z \le 3$ the angle $\frac{\pi}{4}$ around the *x*-axes.

You can plot the above surface with DPGraph if you use the function one $\text{one}(z) = \begin{cases} 1 \textit{ if } 0 \le z \le 1 \\ \text{undefined } \textit{else} \end{cases}$

graph3d(one(z/3)*z=x^2+y^2)

gives the surface $z = x^2 + y^2, z \le 3$.

You obtain the rotated surface if you

1. perform the matrix multiplication $R_x(-\pi/4)\begin{bmatrix} x \\ y \\ z \end{bmatrix}$
2. replace *y* and *z* in **one(z/3)*z=x^2+y^2** by the expressions for the *y*- and *z*-coordinates you obtain for the rotated point in 1.

The below screen shot indicates how to use DERIVE to generate the equation of the rotated surface.

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#7:   TimesOperator := Asterisk
#8:   one(z) :=
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$$\#9: \quad \text{one}\left(\frac{z}{3}\right)*z = x^2 + y^2$$

$$\#10: \quad RX\left(-\frac{\pi}{4}\right)*\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\#11: \quad \begin{bmatrix} x \\ 0.707106*y + 0.707106*z \\ 0.707106*z - 0.707106*y \end{bmatrix}$$

$$\#12: \quad \text{one}\left(\frac{0.707106*z - 0.707106*y}{3}\right)*(0.707106*z - 0.707106*y) = x^2 + (0.707106*y + 0.707106*z)^2$$

Example: Plot the surface you obtain if you rotate the surface $z = x^2 + y^2, z \le 2.5$ the angle $\frac{\pi}{4}$ around the *x*-axes and then the angle *time* around the *z*-axes.

The inverse function of $R_z(time) \circ R_x(\pi/4)$ is $R_x(-\pi/4) \circ R_z(-time)$. Hence, we have to perform the matrix multiplication $R_x(-\pi/4) \circ R_z(-time)\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and then replace *x*, *y* and *z* in $\text{one}(z/2.5) = x^2 + y^2$, by the *x*, *y* and *z*-coordinates you expressions you obtain for the rotated point. You can look at the result in rotpar3.dpg.